

# NONTRIVIAL TWISTED SUMS OF $c_0$ AND $C(K)$

CLAUDIA CORREA AND DANIEL V. TAUSK

ABSTRACT. We obtain a new class of compact Hausdorff spaces  $K$  for which  $c_0$  can be nontrivially twisted with  $C(K)$ .

## 1. INTRODUCTION

In this article, we present a broad new class of compact Hausdorff spaces  $K$  such that there exists a nontrivial twisted sum of  $c_0$  and  $C(K)$ , where  $C(K)$  denotes the Banach space of continuous real-valued functions on  $K$  endowed with the supremum norm. By a *twisted sum* of the Banach spaces  $Y$  and  $X$  we mean a short exact sequence  $0 \rightarrow Y \rightarrow Z \rightarrow X \rightarrow 0$ , where  $Z$  is a Banach space and the maps are bounded linear operators. This twisted sum is called *trivial* if the exact sequence splits, i.e., if the map  $Y \rightarrow Z$  admits a bounded linear left inverse (equivalently, if the map  $Z \rightarrow X$  admits a bounded linear right inverse). In other words, the twisted sum is trivial if the range of the map  $Y \rightarrow Z$  is complemented in  $Z$ ; in this case,  $Z \cong X \oplus Y$ . As in [7], we denote by  $\text{Ext}(X, Y)$  the set of equivalence classes of twisted sums of  $Y$  and  $X$  and we write  $\text{Ext}(X, Y) = 0$  if every such twisted sum is trivial.

Many problems in Banach space theory are related to the quest for conditions under which  $\text{Ext}(X, Y) = 0$ . For instance, an equivalent statement for the classical Theorem of Sobczyk ([5, 13]) is that if  $X$  is a separable Banach space, then  $\text{Ext}(X, c_0) = 0$  ([3, Proposition 3.2]). The converse of the latter statement clearly does not hold in general: for example,  $\text{Ext}(\ell_1(I), c_0) = 0$ , since  $\ell_1(I)$  is a projective Banach space. However, the following question remains open: *is it true that  $\text{Ext}(C(K), c_0) \neq 0$  for any nonseparable  $C(K)$  space?* This problem was stated in [4, 5] and further studied in the recent article [6], in which the author proves that, under the *continuum hypothesis* (CH), the space  $\text{Ext}(C(K), c_0)$  is nonzero for a nonmetrizable compact Hausdorff space  $K$  of finite height. In addition to this result, everything else that is known about the problem is summarized in [6, Proposition 2], namely that  $\text{Ext}(C(K), c_0)$  is nonzero for a  $C(K)$  space under any one of the following assumptions:

---

*Date:* October 22nd, 2015.

*2010 Mathematics Subject Classification.* 46B20, 46E15.

*Key words and phrases.* Banach spaces of continuous functions; twisted sums of Banach spaces; Valdivia compacta.

The first author is sponsored by FAPESP (Process no. 2014/00848-2).

- $K$  is a nonmetrizable Eberlein compact space;
- $K$  is a Valdivia compact space which does not satisfy the countable chain condition (ccc);
- the weight of  $K$  is equal to  $\omega_1$  and the dual space of  $C(K)$  is not weak\*-separable;
- $K$  has the extension property ([8]) and it does not have ccc;
- $C(K)$  contains an isomorphic copy of  $\ell_\infty$ .

Note also that if  $\text{Ext}(Y, c_0) \neq 0$  and  $X$  contains a *complemented* isomorphic copy of  $Y$ , then  $\text{Ext}(X, c_0) \neq 0$ .

Here is an overview of the main results of this article. Theorem 2.3 gives a condition involving biorthogonal systems in a Banach space  $X$  which implies that  $\text{Ext}(X, c_0) \neq 0$ . In the rest of Section 2, we discuss some of its implications when  $X$  is of the form  $C(K)$ . It is proven that if  $K$  contains a homeomorphic copy of  $[0, \omega] \times [0, \mathfrak{c}]$  or of  $2^\mathfrak{c}$ , then  $\text{Ext}(C(K), c_0)$  is nonzero, where  $\mathfrak{c}$  denotes the cardinality of the *continuum*. In Sections 3 and 4, we investigate the consequences of the results of Section 2 for Valdivia and Corson compacta. Recall that Valdivia compact spaces constitute a large superclass of Corson compact spaces closed under arbitrary products; moreover, every Eberlein compact is a Corson compact (see [11] for a survey on Valdivia compacta). Section 3 is devoted to the proof that, under CH, it holds that  $\text{Ext}(C(K), c_0) \neq 0$  for every nonmetrizable Corson compact space  $K$ . The question of whether  $\text{Ext}(C(K), c_0) \neq 0$  for an arbitrary nonmetrizable Valdivia compact space  $K$  remains open (even under CH), but in Section 4 we solve some particular cases of this problem.

## 2. GENERAL RESULTS

Throughout the paper, the weight and the density character of a topological space  $\mathcal{X}$  are denoted, respectively, by  $w(\mathcal{X})$  and  $\text{dens}(\mathcal{X})$ . Moreover, we always denote by  $\chi_A$  the characteristic function of a set  $A$  and by  $|A|$  the cardinality of  $A$ . We start with a technical lemma which is the heart of the proof of Theorem 2.3. A family of sets  $(A_i)_{i \in I}$  is said to be *almost disjoint* if each  $A_i$  is infinite and  $A_i \cap A_j$  is finite, for all  $i, j \in I$  with  $i \neq j$ .

**Lemma 2.1.** *There exists an almost disjoint family  $(A_{n,\alpha})_{n \in \omega, \alpha \in \mathfrak{c}}$  of subsets of  $\omega$  satisfying the following property: for every family  $(A'_{n,\alpha})_{n \in \omega, \alpha \in \mathfrak{c}}$  with each  $A'_{n,\alpha} \subset A_{n,\alpha}$  cofinite in  $A_{n,\alpha}$ , it holds that  $\sup_{p \in \omega} |M_p| = +\infty$ , where:*

$$M_p = \{n \in \omega : p \in \bigcup_{\alpha \in \mathfrak{c}} A'_{n,\alpha}\}.$$

*Proof.* We will obtain an almost disjoint family  $(A_{n,\alpha})_{n \in \omega, \alpha \in \mathfrak{c}}$  of subsets of  $2^{<\omega}$  with the desired property, where  $2^{<\omega} = \bigcup_{k \in \omega} 2^k$  denotes the set of finite sequences in  $2 = \{0, 1\}$ . For each  $\epsilon \in 2^\omega$ , we set:

$$\mathcal{A}_\epsilon = \{\epsilon|_k : k \in \omega\},$$

so that  $(\mathcal{A}_\epsilon)_{\epsilon \in 2^\omega}$  is an almost disjoint family of subsets of  $2^{<\omega}$ . Let  $(\mathcal{B}_\alpha)_{\alpha \in \mathfrak{c}}$  be an enumeration of the uncountable Borel subsets of  $2^\omega$ . Recalling that

$|\mathcal{B}_\alpha| = \mathfrak{c}$  for all  $\alpha \in \mathfrak{c}$  ([12, Theorem 13.6]), one easily obtains by transfinite recursion a family  $(\epsilon_{n,\alpha})_{n \in \omega, \alpha \in \mathfrak{c}}$  of pairwise distinct elements of  $2^\omega$  such that  $\epsilon_{n,\alpha} \in \mathcal{B}_\alpha$ , for all  $n \in \omega$ ,  $\alpha \in \mathfrak{c}$ . Set  $A_{n,\alpha} = \mathcal{A}_{\epsilon_{n,\alpha}}$  and let  $(A'_{n,\alpha})_{n \in \omega, \alpha \in \mathfrak{c}}$  be as in the statement of the lemma. For  $n \in \omega$ , denote by  $D_n$  the set of those  $\epsilon \in 2^\omega$  such that  $n \in M_p$  for all but finitely many  $p \in \mathcal{A}_\epsilon$ . Note that:

$$D_n = \bigcup_{k_0 \in \omega} \bigcap_{k \geq k_0} \bigcup \{C_p : p \in 2^k \text{ with } n \in M_p\},$$

where  $C_p$  denotes the clopen subset of  $2^\omega$  consisting of the extensions of  $p$ . The above equality implies that  $D_n$  is an  $F_\sigma$  (and, in particular, a Borel) subset of  $2^\omega$ . We claim that the complement of  $D_n$  in  $2^\omega$  is countable. Namely, if it were uncountable, there would exist  $\alpha \in \mathfrak{c}$  with  $\mathcal{B}_\alpha = 2^\omega \setminus D_n$ . But, since  $n \in M_p$  for all  $p \in A'_{n,\alpha}$ , we have that  $\epsilon_{n,\alpha} \in D_n$ , contradicting the fact that  $\epsilon_{n,\alpha} \in \mathcal{B}_\alpha$  and proving the claim. To conclude the proof of the lemma, note that for each  $n \geq 1$  the intersection  $\bigcap_{i < n} D_i$  is nonempty; for  $\epsilon \in \bigcap_{i < n} D_i$ , we have that  $\{i : i < n\} \subset M_p$ , for all but finitely many  $p \in \mathcal{A}_\epsilon$ .  $\square$

Let  $X$  be a Banach space. Recall that a *biorthogonal system* in  $X$  is a family  $(x_i, \gamma_i)_{i \in I}$  with  $x_i \in X$ ,  $\gamma_i \in X^*$ ,  $\gamma_i(x_i) = 1$  and  $\gamma_i(x_j) = 0$  for  $i \neq j$ . The *cardinality* of the biorthogonal system  $(x_i, \gamma_i)_{i \in I}$  is defined as the cardinality of  $I$ .

**Definition 2.2.** Let  $(x_i, \gamma_i)_{i \in I}$  be a biorthogonal system in a Banach space  $X$ . We call  $(x_i, \gamma_i)_{i \in I}$  *bounded* if  $\sup_{i \in I} \|x_i\| < +\infty$  and  $\sup_{i \in I} \|\gamma_i\| < +\infty$ ; *weak\*-null* if  $(\gamma_i)_{i \in I}$  is a weak\*-null family, i.e., if  $(\gamma_i(x))_{i \in I}$  is in  $c_0(I)$ , for all  $x \in X$ .

**Theorem 2.3.** Let  $X$  be a Banach space. Assume that there exist a weak\*-null biorthogonal system  $(x_{n,\alpha}, \gamma_{n,\alpha})_{n \in \omega, \alpha \in \mathfrak{c}}$  in  $X$  and a constant  $C \geq 0$  such that:

$$\left\| \sum_{i=1}^k x_{n_i, \alpha_i} \right\| \leq C,$$

for all  $n_1, \dots, n_k \in \omega$  pairwise distinct, all  $\alpha_1, \dots, \alpha_k \in \mathfrak{c}$ , and all  $k \geq 1$ . Then  $\text{Ext}(X, c_0) \neq 0$ .

*Proof.* By [7, Proposition 1.4.f], we have that  $\text{Ext}(X, c_0) = 0$  if and only if every bounded operator  $T : X \rightarrow \ell_\infty/c_0$  admits a *lifting*<sup>1</sup>, i.e., a bounded operator  $\widehat{T} : X \rightarrow \ell_\infty$  with  $T(x) = \widehat{T}(x) + c_0$ , for all  $x \in X$ . Let us then show that there exists an operator  $T : X \rightarrow \ell_\infty/c_0$  that does not admit a lifting. To this aim, let  $(A_{n,\alpha})_{n \in \omega, \alpha \in \mathfrak{c}}$  be an almost disjoint family as in Lemma 2.1 and consider the unique isometric embedding  $S : c_0(\omega \times \mathfrak{c}) \rightarrow \ell_\infty/c_0$  such that  $S(e_{n,\alpha}) = \chi_{A_{n,\alpha}} + c_0$ , where  $(e_{n,\alpha})_{n \in \omega, \alpha \in \mathfrak{c}}$  denotes the canonical basis of

<sup>1</sup>More concretely, a nontrivial twisted sum of  $c_0$  and  $X$  is obtained by considering the pull-back of the short exact sequence  $0 \rightarrow c_0 \rightarrow \ell_\infty \rightarrow \ell_\infty/c_0 \rightarrow 0$  by an operator  $T : X \rightarrow \ell_\infty/c_0$  that does not admit a lifting.

$c_0(\omega \times \mathfrak{c})$ . Denote by  $\Gamma : X \rightarrow c_0(\omega \times \mathfrak{c})$  the bounded operator with coordinate functionals  $(\gamma_{n,\alpha})_{n \in \omega, \alpha \in \mathfrak{c}}$  and set  $T = S \circ \Gamma : X \rightarrow \ell_\infty/c_0$ . Assuming by contradiction that there exists a lifting  $\widehat{T}$  of  $T$  and denoting by  $(\mu_p)_{p \in \omega}$  the sequence of coordinate functionals of  $\widehat{T}$ , we have that the set:

$$A'_{n,\alpha} = \{p \in A_{n,\alpha} : \mu_p(x_{n,\alpha}) \geq \frac{1}{2}\}$$

is cofinite in  $A_{n,\alpha}$ . It follows that for each  $k \geq 1$ , there exist  $p \in \omega$ ,  $n_1, \dots, n_k \in \omega$  pairwise distinct, and  $\alpha_1, \dots, \alpha_k \in \mathfrak{c}$  such that  $p \in A'_{n_i, \alpha_i}$ , for  $i = 1, \dots, k$ . Hence:

$$\frac{k}{2} \leq \mu_p\left(\sum_{i=1}^k x_{n_i, \alpha_i}\right) \leq C\|\widehat{T}\|,$$

which yields a contradiction.  $\square$

**Corollary 2.4.** *Let  $K$  be a compact Hausdorff space. Assume that there exists a bounded weak\*-null biorthogonal system  $(f_{n,\alpha}, \gamma_{n,\alpha})_{n \in \omega, \alpha \in \mathfrak{c}}$  in  $C(K)$  such that  $f_{n,\alpha} f_{m,\beta} = 0$ , for all  $n, m \in \omega$  with  $n \neq m$  and all  $\alpha, \beta \in \mathfrak{c}$ . Then  $\text{Ext}(C(K), c_0) \neq 0$ .*  $\square$

**Definition 2.5.** We say that a compact Hausdorff space  $K$  satisfies *property (\*)* if there exist a sequence  $(F_n)_{n \in \omega}$  of closed subsets of  $K$  and a bounded weak\*-null biorthogonal system  $(f_{n,\alpha}, \gamma_{n,\alpha})_{n \in \omega, \alpha \in \mathfrak{c}}$  in  $C(K)$  such that:

$$(1) \quad F_n \cap \overline{\bigcup_{m \neq n} F_m} = \emptyset$$

and  $\text{supp } f_{n,\alpha} \subset F_n$ , for all  $n \in \omega$  and all  $\alpha \in \mathfrak{c}$ , where  $\text{supp } f_{n,\alpha}$  denotes the support of  $f_{n,\alpha}$ .

In what follows, we denote by  $M(K)$  the space of finite countably-additive signed regular Borel measures on  $K$ , endowed with the total variation norm. We identify as usual the dual space of  $C(K)$  with  $M(K)$ .

**Lemma 2.6.** *Let  $K$  be a compact Hausdorff space and  $L$  be a closed subspace of  $K$ . If  $L$  satisfies property (\*), then so does  $K$ .*

*Proof.* Consider, as in Definition 2.5, a sequence  $(F_n)_{n \in \omega}$  of closed subsets of  $L$  and a bounded weak\*-null biorthogonal system  $(f_{n,\alpha}, \gamma_{n,\alpha})_{n \in \omega, \alpha \in \mathfrak{c}}$  in  $C(L)$ . By recursion on  $n$ , one easily obtains a sequence  $(U_n)_{n \in \omega}$  of pairwise disjoint open subsets of  $K$  with each  $U_n$  containing  $F_n$ . Let  $V_n$  be an open subset of  $K$  with  $F_n \subset V_n \subset \overline{V_n} \subset U_n$ . Using Tietze's Extension Theorem and Urysohn's Lemma, we get a continuous extension  $\tilde{f}_{n,\alpha}$  of  $f_{n,\alpha}$  to  $K$  with support contained in  $\overline{V_n}$  and having the same norm as  $f_{n,\alpha}$ . To conclude the proof, let  $\tilde{\gamma}_{n,\alpha} \in M(K)$  be the extension of  $\gamma_{n,\alpha} \in M(L)$  that vanishes identically outside of  $L$  and observe that  $(\tilde{f}_{n,\alpha}, \tilde{\gamma}_{n,\alpha})_{n \in \omega, \alpha \in \mathfrak{c}}$  is a bounded weak\*-null biorthogonal system in  $C(K)$ .  $\square$

As an immediate consequence of Lemma 2.6 and Corollary 2.4, we obtain the following result.

**Theorem 2.7.** *If a compact Hausdorff space  $L$  satisfies property  $(*)$ , then every compact Hausdorff space  $K$  containing a homeomorphic copy of  $L$  satisfies  $\text{Ext}(C(K), c_0) \neq 0$ .  $\square$*

We now establish a few results which give sufficient conditions for a space  $K$  to satisfy property  $(*)$ . Recall that, given a closed subset  $F$  of a compact Hausdorff space  $K$ , an *extension operator* for  $F$  in  $K$  is a bounded operator  $E : C(F) \rightarrow C(K)$  which is a right inverse for the restriction operator  $C(K) \ni f \mapsto f|_F \in C(F)$ . Note that  $F$  admits an extension operator in  $K$  if and only if the kernel

$$C(K|F) = \{f \in C(K) : f|_F = 0\}$$

of the restriction operator is complemented in  $C(K)$ . A point  $x$  of a topological space  $\mathcal{X}$  is called a *cluster point* of a sequence  $(S_n)_{n \in \omega}$  of subsets of  $\mathcal{X}$  if every neighborhood of  $x$  intersects  $S_n$  for infinitely many  $n \in \omega$ .

**Lemma 2.8.** *Let  $K$  be a compact Hausdorff space. Assume that there exist a sequence  $(F_n)_{n \in \omega}$  of pairwise disjoint closed subsets of  $K$  and a closed subset  $F$  of  $K$  satisfying the following conditions:*

- (a)  *$F$  admits an extension operator in  $K$ ;*
- (b) *every cluster point of  $(F_n)_{n \in \omega}$  is in  $F$  and  $F_n \cap F = \emptyset$ , for all  $n \in \omega$ ;*
- (c) *there exists a family  $(f_{n,\alpha}, \gamma_{n,\alpha})_{n \in \omega, \alpha \in \mathfrak{c}}$ , where  $(f_{n,\alpha}, \gamma_{n,\alpha})_{\alpha \in \mathfrak{c}}$  is a weak\*-null biorthogonal system in  $C(F_n)$  for each  $n \in \omega$  and*

$$\sup_{n \in \omega, \alpha \in \mathfrak{c}} \|f_{n,\alpha}\| < +\infty, \quad \sup_{n \in \omega, \alpha \in \mathfrak{c}} \|\gamma_{n,\alpha}\| < +\infty.$$

*Then  $K$  satisfies property  $(*)$ .*

*Proof.* From (b) and the fact that the  $F_n$  are pairwise disjoint it follows that (1) holds. Let  $(U_n)_{n \in \omega}$ ,  $(V_n)_{n \in \omega}$ , and  $(\tilde{f}_{n,\alpha})_{n \in \omega, \alpha \in \mathfrak{c}}$  be as in the proof of Lemma 2.6; we assume also that  $\overline{V_n} \cap F = \emptyset$ , for all  $n \in \omega$ . Let  $\tilde{\gamma}_{n,\alpha} \in M(K)$  be the extension of  $\gamma_{n,\alpha} \in M(F_n)$  that vanishes identically outside of  $F_n$ . We have that  $(\tilde{f}_{n,\alpha}, \tilde{\gamma}_{n,\alpha})_{n \in \omega, \alpha \in \mathfrak{c}}$  is a bounded biorthogonal system in  $C(K)$  and that  $(\tilde{\gamma}_{n,\alpha})_{\alpha \in \mathfrak{c}}$  is weak\*-null for each  $n$ , though it is not true in general that the entire family  $(\tilde{\gamma}_{n,\alpha})_{n \in \omega, \alpha \in \mathfrak{c}}$  is weak\*-null. In order to take care of this problem, let  $P : C(K) \rightarrow C(K|F)$  be a bounded projection and set  $\hat{\gamma}_{n,\alpha} = \tilde{\gamma}_{n,\alpha} \circ P$ . Since all  $\tilde{f}_{n,\alpha}$  are in  $C(K|F)$ , we have that  $(\tilde{f}_{n,\alpha}, \hat{\gamma}_{n,\alpha})_{n \in \omega, \alpha \in \mathfrak{c}}$  is biorthogonal. To prove that  $(\hat{\gamma}_{n,\alpha})_{n \in \omega, \alpha \in \mathfrak{c}}$  is weak\*-null, note that (b) implies that  $\lim_{n \rightarrow +\infty} \|f|_{F_n}\| = 0$ , for all  $f \in C(K|F)$ .  $\square$

**Corollary 2.9.** *Let  $K$  be a compact Hausdorff space. If  $C(K)$  admits a bounded weak\*-null biorthogonal system of cardinality  $\mathfrak{c}$ , then the space  $[0, \omega] \times K$  satisfies property  $(*)$ . In particular,  $L \times K$  satisfies property  $(*)$  for every compact Hausdorff space  $L$  containing a nontrivial convergent sequence.*

*Proof.* Take  $F_n = \{n\} \times K$ ,  $F = \{\omega\} \times K$ , and use the fact that  $F$  is a retract of  $[0, \omega] \times K$  and thus admits an extension operator in  $[0, \omega] \times K$ .  $\square$

**Corollary 2.10.** *The spaces  $[0, \omega] \times [0, \mathfrak{c}]$  and  $2^{\mathfrak{c}}$  satisfy property (\*). In particular, a product of at least  $\mathfrak{c}$  compact Hausdorff spaces with more than one point satisfies property (\*).*

*Proof.* The family  $(\chi_{[0, \alpha]}, \delta_\alpha - \delta_{\alpha+1})_{\alpha \in \mathfrak{c}}$  is a bounded weak\*-null biorthogonal system in  $C([0, \mathfrak{c}])$ , where  $\delta_\alpha \in M([0, \mathfrak{c}])$  denotes the probability measure with support  $\{\alpha\}$ . It follows from Corollary 2.9 that  $[0, \omega] \times [0, \mathfrak{c}]$  satisfies property (\*). To see that  $2^{\mathfrak{c}}$  also does, note that the map  $[0, \mathfrak{c}] \ni \alpha \mapsto \chi_\alpha \in 2^{\mathfrak{c}}$  embeds  $[0, \mathfrak{c}]$  into  $2^{\mathfrak{c}}$ , so that  $2^{\mathfrak{c}} \cong 2^\omega \times 2^{\mathfrak{c}}$  contains a homeomorphic copy of  $[0, \omega] \times [0, \mathfrak{c}]$ .  $\square$

Recall that a *projectional resolution of the identity* (PRI) of a Banach space  $X$  is a family  $(P_\alpha)_{\omega \leq \alpha \leq \text{dens}(X)}$  of projection operators  $P_\alpha : X \rightarrow X$  satisfying the following conditions:

- $\|P_\alpha\| = 1$ , for  $\omega \leq \alpha \leq \text{dens}(X)$ ;
- $P_{\text{dens}(X)}$  is the identity of  $X$ ;
- $P_\alpha[X] \subset P_\beta[X]$  and  $\text{Ker}(P_\beta) \subset \text{Ker}(P_\alpha)$ , for  $\omega \leq \alpha \leq \beta \leq \text{dens}(X)$ ;
- $P_\alpha(x) = \lim_{\beta < \alpha} P_\beta(x)$ , for all  $x \in X$ , if  $\omega < \alpha \leq \text{dens}(X)$  is a limit ordinal;
- $\text{dens}(P_\alpha[X]) \leq |\alpha|$ , for  $\omega \leq \alpha \leq \text{dens}(X)$ .

We call the PRI *strictly increasing* if  $P_\alpha[X]$  is a proper subspace of  $P_\beta[X]$ , for  $\omega \leq \alpha < \beta \leq \text{dens}(X)$ .

**Corollary 2.11.** *Let  $K$  and  $L$  be compact Hausdorff spaces such that  $L$  contains a nontrivial convergent sequence and  $w(K) \geq \mathfrak{c}$ . If  $C(K)$  admits a strictly increasing PRI, then the space  $L \times K$  satisfies property (\*).*

*Proof.* This follows from Corollary 2.9 by observing that if a Banach space  $X$  admits a strictly increasing PRI, then  $X$  admits a weak\*-null biorthogonal system  $(x_\alpha, \gamma_\alpha)_{\omega \leq \alpha < \text{dens}(X)}$  with  $\|x_\alpha\| = 1$  and  $\|\gamma_\alpha\| \leq 2$ , for all  $\alpha$ . Namely, pick a unit vector  $x_\alpha$  in  $P_{\alpha+1}[X] \cap \text{Ker}(P_\alpha)$  and set  $\gamma_\alpha = \phi_\alpha \circ (P_{\alpha+1} - P_\alpha)$ , where  $\phi_\alpha \in X^*$  is a norm-one functional satisfying  $\phi_\alpha(x_\alpha) = 1$ .  $\square$

### 3. NONTRIVIAL TWISTED SUMS FOR CORSON COMPACTA

Let us recall some standard definitions. Given an index set  $I$ , we write  $\Sigma(I) = \{x \in \mathbb{R}^I : \text{supp } x \text{ is countable}\}$ , where the support  $\text{supp } x$  of  $x$  is defined by  $\text{supp } x = \{i \in I : x_i \neq 0\}$ . Given a compact Hausdorff space  $K$ , we call  $A$  a  $\Sigma$ -subset of  $K$  if there exist an index set  $I$  and a continuous injection  $\varphi : K \rightarrow \mathbb{R}^I$  such that  $A = \varphi^{-1}[\Sigma(I)]$ . The space  $K$  is called a *Valdivia compactum* if it admits a dense  $\Sigma$ -subset and it is called a *Corson compactum* if  $K$  is a  $\Sigma$ -subset of itself. This section will be dedicated to the proof of the following result.

**Theorem 3.1.** *If  $K$  is a Corson compact space with  $w(K) \geq \mathfrak{c}$ , then  $\text{Ext}(C(K), c_0) \neq 0$ . In particular, under CH, we have  $\text{Ext}(C(K), c_0) \neq 0$  for any nonmetrizable Corson compact space  $K$ .*

The fact that  $\text{Ext}(C(K), c_0) \neq 0$  for a Valdivia compact space  $K$  which does not have ccc is already known ([6, Proposition 2]). Our strategy for the proof of Theorem 3.1 is to use Lemma 2.8 to show that if  $K$  is a Corson compact space with  $w(K) \geq \mathfrak{c}$  having ccc, then  $K$  satisfies property (\*). We start with a lemma that will be used as a tool for verifying the assumptions of Lemma 2.8. Recall that a closed subset of a topological space is called *regular* if it is the closure of an open set (equivalently, if it is the closure of its own interior). Obviously, a closed subset of a Corson compact space is again Corson and a regular closed subset of a Valdivia compact space is again Valdivia.

**Lemma 3.2.** *Let  $K$  be a compact Hausdorff space and  $F$  be a closed non-open  $G_\delta$  subset of  $K$ . Then there exists a sequence  $(F_n)_{n \in \omega}$  of nonempty pairwise disjoint regular closed subsets of  $K$  such that condition (b) in the statement of Lemma 2.8 holds.*

*Proof.* We can write  $F = \bigcap_{n \in \omega} V_n$ , with each  $V_n$  open in  $K$  and  $\overline{V_{n+1}}$  properly contained in  $V_n$ . Set  $U_n = V_n \setminus \overline{V_{n+1}}$ , so that all cluster points of  $(U_n)_{n \in \omega}$  are in  $F$ . To conclude the proof, let  $F_n$  be a nonempty regular closed set contained in  $U_n$ .  $\square$

Once we get the closed sets  $(F_n)_{n \in \omega}$  from Lemma 3.2, we still have to verify the rest of the conditions in the statement of Lemma 2.8. First, we need an assumption ensuring that  $w(F_n) \geq \mathfrak{c}$ , for all  $n$ . To this aim, given a point  $x$  of a topological space  $\mathcal{X}$ , we define the *weight of  $x$  in  $\mathcal{X}$*  by:

$$w(x, \mathcal{X}) = \min \{w(V) : V \text{ neighborhood of } x \text{ in } \mathcal{X}\}.$$

Recall that if  $K$  is a Valdivia compact space, then  $C(K)$  admits a PRI ([15, Theorem 2]). Moreover, a trivial adaptation of the proof in [15] shows in fact that  $C(K)$  admits a strictly increasing PRI. Thus, by the argument in the proof of Corollary 2.11,  $C(K)$  admits a weak\*-null biorthogonal system  $(f_\alpha, \gamma_\alpha)_{\omega \leq \alpha < w(K)}$  such that  $\|f_\alpha\| \leq 1$  and  $\|\gamma_\alpha\| \leq 2$ , for all  $\alpha$ . The following result is now immediately obtained.

**Corollary 3.3.** *Let  $K$  be a Valdivia compact space such that  $w(x, K) \geq \mathfrak{c}$ , for all  $x \in K$ . Assume that there exists a closed non-open  $G_\delta$  subset  $F$  admitting an extension operator in  $K$ . Then  $K$  satisfies property (\*).  $\square$*

Assuming that  $K$  has ccc, the next lemma allows us to reduce the proof of Theorem 3.1 to the case when  $w(x, K) \geq \mathfrak{c}$ , for all  $x \in K$ .

**Lemma 3.4.** *Let  $K$  be a ccc Valdivia compact space and set:*

$$H = \{x \in K : w(x, K) \geq \mathfrak{c}\}.$$

*Then:*

- (a)  $H \neq \emptyset$ , if  $w(K) \geq \mathfrak{c}$ ;
- (b)  $w(K \setminus \text{int}(H)) < \mathfrak{c}$ , where  $\text{int}(H)$  denotes the interior of  $H$ ;
- (c)  $H$  is a regular closed subset of  $K$ ;

(d)  $w(x, H) \geq \mathfrak{c}$ , for all  $x \in H$ .

*Proof.* If  $H = \emptyset$ , then  $K$  can be covered by a finite number of open sets with weight less than  $\mathfrak{c}$ , so that  $w(K) < \mathfrak{c}$ . This proves (a). To prove (b), let  $(U_i)_{i \in I}$  be maximal among antichains of open subsets of  $K$  with weight less than  $\mathfrak{c}$ . Since  $I$  is countable and  $\mathfrak{c}$  has uncountable cofinality, we have that  $U = \bigcup_{i \in I} U_i$  has weight less than  $\mathfrak{c}$ . From the maximality of  $(U_i)_{i \in I}$ , it follows that  $K \setminus H \subset \overline{U}$ ; then  $K \setminus \text{int}(H) = \overline{K \setminus H} \subset \overline{U}$ . To conclude the proof of (b), let us show that  $w(\overline{U}) < \mathfrak{c}$ . Let  $A$  be a dense  $\Sigma$ -subset of  $K$  and let  $D$  be a dense subset of  $A \cap U$  with  $|D| \leq w(U)$ . Then  $\overline{D}$  is homeomorphic to a subspace of  $\mathbb{R}^{w(U)}$ , so that  $w(\overline{U}) = w(\overline{D}) \leq w(U) < \mathfrak{c}$ . To prove (c), note that  $H$  is clearly closed; moreover, by (b), the open set  $K \setminus \text{int}(H)$  has weight less than  $\mathfrak{c}$  and hence it is contained in  $K \setminus H$ . Finally, to prove (d), let  $V$  be a closed neighborhood in  $K$  of some  $x \in H$ . By (b), we have  $w(V \setminus H) < \mathfrak{c}$ . Recall from [9, p. 26] that if a compact Hausdorff space is the union of not more than  $\kappa$  subsets of weight not greater than  $\kappa$ , then the weight of the space is not greater than  $\kappa$ . Since  $w(V) \geq \mathfrak{c}$ , it follows from such result that  $w(V \cap H) \geq \mathfrak{c}$ .  $\square$

*Proof of Theorem 3.1.* By Lemma 3.4, it suffices to prove that if  $K$  is a nonempty Corson compact space such that  $w(x, K) \geq \mathfrak{c}$  for all  $x \in K$ , then  $K$  satisfies property (\*). Since a nonempty Corson compact space  $K$  admits a  $G_\delta$  point  $x$  ([11, Theorem 3.3]), this fact follows from Corollary 3.3 with  $F = \{x\}$ .  $\square$

*Remark 3.5.* It is known that under Martin's Axiom (MA) and the negation of CH, every ccc Corson compact is metrizable ([1]). Thus, Theorem 3.1 implies that  $\text{Ext}(C(K), c_0) \neq 0$  for every nonmetrizable Corson compact space  $K$  under MA.

#### 4. TOWARDS THE GENERAL VALDIVIA CASE

In this section we prove that  $\text{Ext}(C(K), c_0) \neq 0$  for certain classes of nonmetrizable Valdivia compact spaces  $K$  and we propose a strategy for dealing with the general problem. First, let us state some results which are immediate consequences of what we have done so far.

**Proposition 4.1.** *If  $K$  is a Valdivia compact space with  $w(K) \geq \mathfrak{c}$  and  $L$  is a compact Hausdorff space containing a nontrivial convergent sequence, then  $L \times K$  satisfies property (\*).*

*Proof.* As we have observed in Section 3, if  $K$  is a Valdivia compact space, then  $C(K)$  admits a strictly increasing PRI. The conclusion follows from Corollary 2.11.  $\square$

**Proposition 4.2.** *Let  $K$  be a Valdivia compact space admitting a  $G_\delta$  point  $x$  with  $w(x, K) \geq \mathfrak{c}$ . Then  $\text{Ext}(C(K), c_0) \neq 0$  and, if  $K$  has ccc, then  $K$  satisfies property (\*).*



*Proof.* As mentioned before, the non-ccc case is already known. Assuming that  $K$  has ccc, define  $H$  as in Lemma 3.4 and conclude that  $H$  satisfies property  $(*)$  using Corollary 3.3 with  $F = \{x\}$ .  $\square$

**Corollary 4.3.** *Let  $K$  be a Valdivia compact space with  $w(K) \geq \mathfrak{c}$  admitting a dense  $\Sigma$ -subset  $A$  such that  $K \setminus A$  is of first category. Then  $\text{Ext}(C(K), c_0) \neq 0$  and, if  $K$  has ccc, then  $K$  satisfies property  $(*)$ .*

*Proof.* By [11, Theorem 3.3],  $K$  has a dense subset of  $G_\delta$  points. Assuming that  $K$  has ccc and defining  $H$  as in Lemma 3.4, we obtain that  $H$  contains a  $G_\delta$  point of  $K$ , which implies that  $K$  satisfies the assumptions of Proposition 4.2.  $\square$

Now we investigate conditions under which a Valdivia compact space  $K$  contains a homeomorphic copy of  $[0, \omega] \times [0, \mathfrak{c}]$ . Given an index set  $I$  and a subset  $J$  of  $I$ , we denote by  $r_J : \mathbb{R}^I \rightarrow \mathbb{R}^I$  the map defined by setting  $r_J(x)|_J = x|_J$  and  $r_J(x)|_{I \setminus J} \equiv 0$ , for all  $x \in \mathbb{R}^I$ . Following [2], given a subset  $K$  of  $\mathbb{R}^I$ , we say that  $J \subset I$  is  $K$ -good if  $r_J[K] \subset K$ . In [2, Lemma 1.2], it is proven that if  $K$  is a compact subset of  $\mathbb{R}^I$  and  $\Sigma(I) \cap K$  is dense in  $K$ , then every infinite subset  $J$  of  $I$  is contained in a  $K$ -good set  $J'$  with  $|J| = |J'|$ .

**Proposition 4.4.** *Let  $K$  be a Valdivia compact space admitting a dense  $\Sigma$ -subset  $A$  such that some point of  $K \setminus A$  is the limit of a nontrivial sequence in  $K$ . Then  $K$  contains a homeomorphic copy of  $[0, \omega] \times [0, \omega_1]$ . In particular, assuming  $CH$ , we have that  $K$  satisfies property  $(*)$ .*

*Proof.* We can obviously assume that  $K$  is a compact subset of some  $\mathbb{R}^I$  and that  $A = \Sigma(I) \cap K$ . Since  $A$  is sequentially closed, our hypothesis implies that there exists a continuous injective map  $[0, \omega] \ni n \mapsto x_n \in K \setminus A$ . Let  $J$  be a countable subset of  $I$  such that  $x_n|_J \neq x_m|_J$ , for all  $n, m \in [0, \omega]$  with  $n \neq m$ . Using [2, Lemma 1.2] and transfinite recursion, one easily obtains a family  $(J_\alpha)_{\alpha \leq \omega_1}$  of  $K$ -good subsets of  $I$  satisfying the following conditions:

- $J_\alpha$  is countable, for  $\alpha < \omega_1$ ;
- $J \subset J_0$ ;
- $J_\alpha \subset J_\beta$ , for  $0 \leq \alpha \leq \beta \leq \omega_1$ ;
- $J_\alpha = \bigcup_{\beta < \alpha} J_\beta$ , for limit  $\alpha \in [0, \omega_1]$ ;
- for all  $n \in [0, \omega]$ , the map  $[0, \omega_1] \ni \alpha \mapsto J_\alpha \cap \text{supp } x_n$  is injective.

Given these conditions, it is readily checked that the map

$$[0, \omega] \times [0, \omega_1] \ni (n, \alpha) \mapsto r_{J_\alpha}(x_n) \in K$$

is continuous and injective.  $\square$

*Remark 4.5.* The following converse of Proposition 4.4 also holds: if  $K$  is a Valdivia compact space containing a homeomorphic copy of  $[0, \omega] \times [0, \omega_1]$ , then  $K \setminus A$  contains a nontrivial convergent sequence, for *any* dense  $\Sigma$ -subset  $A$  of  $K$ . Namely, if  $\phi : [0, \omega] \times [0, \omega_1] \rightarrow K$  is a continuous injection, then  $\phi(\omega, \alpha) \in K \setminus A$  for some  $\alpha \in [0, \omega_1]$ , since  $[0, \omega_1]$  is not Corson ([11, Example 1.10 (i)]). Moreover, the nontrivial sequence  $(\phi(n, \alpha))_{n \in \omega}$  converges

to  $\phi(\omega, \alpha)$ . One consequence of this observation is that if  $K \setminus A$  contains a nontrivial convergent sequence for *some* dense  $\Sigma$ -subset  $A$  of  $K$ , then  $K \setminus A$  contains a nontrivial convergent sequence for *any* dense  $\Sigma$ -subset  $A$  of  $K$ .

*Remark 4.6.* If a Valdivia compact space  $K$  admits two distinct dense  $\Sigma$ -subsets, then the assumption of Proposition 4.4 holds for  $K$ . Namely, given dense  $\Sigma$ -subsets  $A$  and  $B$  of  $K$  and a point  $x \in A \setminus B$ , then  $x$  is not isolated, since  $B$  is dense. Moreover,  $x$  is not isolated in  $A$ , because  $A$  is dense. Finally, since  $A$  is a Fréchet–Urysohn space ([11, Lemma 1.6 (ii)]),  $x$  is the limit of a sequence in  $A \setminus \{x\}$ .

Finally, we observe that the validity of the following conjecture would imply, under CH, that  $\text{Ext}(C(K), c_0) \neq 0$  for any nonmetrizable Valdivia compact space  $K$ .

**Conjecture.** *If  $K$  is a nonempty Valdivia compact space having ccc, then either  $K$  has a  $G_\delta$  point or  $K$  admits a nontrivial convergent sequence in the complement of a dense  $\Sigma$ -subset.*

To see that the conjecture implies the desired result, use Lemma 3.4 and Propositions 4.2 and 4.4, keeping in mind that a regular closed subset of a ccc space has ccc as well. The conjecture remains open, but in what follows we present an example showing that it is false if the assumption that  $K$  has ccc is removed.

Recall that a *tree* is a partially ordered set  $(T, \leq)$  such that, for all  $t \in T$ , the set  $(\cdot, t) = \{s \in T : s < t\}$  is well-ordered. As in [14, p. 288], we define a compact Hausdorff space from a tree  $T$  by considering the subspace  $P(T)$  of  $2^T$  consisting of all characteristic functions of paths of  $T$ ; by a *path* of  $T$  we mean a totally ordered subset  $A$  of  $T$  such that  $(\cdot, t) \subset A$ , for all  $t \in A$ . It is easy to see that  $P(T)$  is closed in  $2^T$ ; we call it the *path space* of  $T$ .

Denote by  $S(\omega_1)$  the set of countable successor ordinals and consider the tree  $T = \bigcup_{\alpha \in S(\omega_1)} \omega_1^\alpha$ , partially ordered by inclusion. The path space  $P(T)$  is the image of the injective map  $\Lambda \ni \lambda \mapsto \chi_{A(\lambda)} \in 2^T$ , where  $\Lambda = \bigcup_{\alpha \leq \omega_1} \omega_1^\alpha$  and  $A(\lambda) = \{t \in T : t \subset \lambda\}$ .

**Proposition 4.7.** *If the tree  $T$  is defined as above, then its path space  $P(T)$  is a compact subspace of  $\mathbb{R}^T$  satisfying the following conditions:*

- (a)  $P(T) \cap \Sigma(T)$  is dense in  $P(T)$ , so that  $P(T)$  is Valdivia;
- (b)  $P(T)$  has no  $G_\delta$  points;
- (c) no point of  $P(T) \setminus \Sigma(T)$  is the limit of a nontrivial sequence in  $P(T)$ .

*Proof.* To prove (a), note that  $\chi_{A(\lambda)} = \lim_{\alpha < \omega_1} \chi_{A(\lambda|_\alpha)}$  for all  $\lambda \in \omega_1^{\omega_1}$ . Let us prove (b). Since  $P(T)$  is Valdivia, every  $G_\delta$  point of  $P(T)$  must be in  $\Sigma(T)$  ([10, Proposition 2.2 (3)]), i.e., it must be of the form  $\chi_{A(\lambda)}$ , with  $\lambda \in \omega_1^\alpha$ ,  $\alpha < \omega_1$ . To see that  $\chi_{A(\lambda)}$  cannot be a  $G_\delta$  point of  $P(T)$ , it suffices to check that for any countable subset  $E$  of  $T$ , there exists  $\mu \in \Lambda$ ,  $\mu \neq \lambda$ , such that  $\chi_{A(\lambda)}$  and  $\chi_{A(\mu)}$  are identical on  $E$ . To this aim, simply take

$\mu = \lambda \cup \{(\alpha, \beta)\}$ , with  $\beta \in \omega_1 \setminus \{t(\alpha) : t \in E \text{ and } \alpha \in \text{dom}(t)\}$ . Finally, to prove (c), let  $(\chi_{A(\lambda_n)})_{n \geq 1}$  be a sequence of pairwise distinct elements of  $P(T)$  converging to some  $\epsilon \in P(T)$  and note that the support of  $\epsilon$  must be contained in the countable set  $\bigcup_{n \neq m} (A(\lambda_n) \cap A(\lambda_m))$ .  $\square$

It is easy to see that, for  $T$  defined as above, the space  $P(T)$  does not have ccc. Namely, setting  $U_t = \{\epsilon \in P(T) : \epsilon(t) = 1\}$  for  $t \in T$ , we have that  $U_t$  is a nonempty open subset of  $P(T)$  and that  $U_t \cap U_s = \emptyset$ , when  $t, s \in T$  are incomparable.

## REFERENCES

- [1] S. A. Argyros, S. Mercourakis, S. Negrepontis, Analytic properties of Corson compact spaces, Proc. Fifth Prague Topological Symposium, Heldermann Verlag, Berlin, 1982, 12—23.
- [2] S. A. Argyros, S. Mercourakis, S. Negrepontis, Functional-analytic properties of Corson compact spaces, *Studia Math.* 89 (1988) 197—229.
- [3] A. Avilés, F. Cabello, J. M. F. Castillo, M. González, Y. Moreno, On separably injective Banach spaces, *Advances Math.* 234 (2013) 192—216.
- [4] F. Cabello, J. M. F. Castillo, N. J. Kalton, D. T. Yost, Twisted sums with  $C(K)$  spaces, *Trans. Amer. Math. Soc.* 355 (11) (2003) 4523—4541.
- [5] F. Cabello, J. M. F. Castillo, D. Yost, Sobczyk's Theorem from A to B, *Extracta Math.* 15 (2) (2000) 391—420.
- [6] J. M. F. Castillo, Nonseparable  $C(K)$ -spaces can be twisted when  $K$  is a finite height compact, to appear in *Topology Appl.*
- [7] J. M. F. Castillo, M. González, Three-space problems in Banach space theory, *Lecture Notes in Mathematics*, vol. 1667, Springer, Berlin, 1997.
- [8] C. Correa, D. V. Tausk, Extension property and complementation of isometric copies of continuous function spaces, *Results Math.* 67 (2015) 445—455.
- [9] R. Hodel, Cardinal functions I, in: K. Kunen, J. E. Vaughan (Eds.), *Handbook of set-theoretic topology*, North-Holland, Amsterdam, 1984.
- [10] O. Kalenda, Continuous images and other topological properties of Valdivia compacta, *Fundamenta Math.* 162 (1999) 181—192.
- [11] O. Kalenda, Valdivia compact spaces in topology and Banach space theory, *Extracta Math.* 15 (1) (2000) 1—85.
- [12] A. S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics, vol. 156, Springer Verlag, New York, 1995.
- [13] A. Sobczyk, Projection of the space  $(m)$  on its subspace  $(c_0)$ , *Bull. Amer. Math. Soc.* 47 (1941) 938—947.
- [14] S. Todorcević, Trees and linearly ordered sets, in: K. Kunen, J. E. Vaughan (Eds.), *Handbook of set-theoretic topology*, North-Holland, Amsterdam, 1984.
- [15] M. Valdivia, Projective resolution of identity in  $C(K)$  spaces, *Arch. Math.* 54 (1990) 493—498.

DEPARTAMENTO DE MATEMÁTICA,  
UNIVERSIDADE DE SÃO PAULO, BRAZIL  
*E-mail address:* claudiac.mat@gmail.com

DEPARTAMENTO DE MATEMÁTICA,  
UNIVERSIDADE DE SÃO PAULO, BRAZIL  
*E-mail address:* tauska@ime.usp.br  
*URL:* <http://www.ime.usp.br/~tauska>